Quantum Chaos in a Billiard System

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A billiard system is one which exists within a bound region containing a particle which experiences complete elastic collisions at the boundary. The potential energy within the region is zero and infinite along the boundary. An example of a billiard system in one dimension would be the infinite square well[4]. Extending to a two dimensional region allows for the analysis of circular and stadium billiard systems. The analysis of a circular billiard system in quantum mechanics is consistent with classical dynamical systems of billiards. Furthermore, the analysis of a stadium billiard system shows that the system exhibits chaotic behaviour. It appears that when the classical dynamics of a system are solvable or chaotic, the quantum counterpart is as well[1]. Future developments regarding quantum chaos involving a two-spinless-fermion system in a quantum wire was also explored. The future development concluded that the classical dynamics had a scaling property, which the quantum counterpart did not have. A method for measuring chaotic irregularity was also found.

I. INTRODUCTION

Quantum Chaos, not unlike classical chaos, is a form of nonlinear dynamics. There is great debate about whether quantum chaos exists or not. There are currently three accepted classifications of quantum chaos: quantised chaos, semi-quantum chaos and true quantum chaos. If quantum chaos is to exist, Hamiltonian chaos could be extended into the quantum regime. In this paper, only Hamiltonian systems are studied as there are no occurrences of attractors nor repellors. Unfortunately, the complexity of their respective dynamics increases as $\mathbb{R}^n$ increases; subsequently, $\mathbb{R}^2$ is the highest dimension dealt with herein. Dynamical systems in classical mechanics can be expressed as billiards. The infinite square well, infinite circular well and a stadium system can all be modelled with billiards. Particles in billiards alternate between motion in a straight line and mirror-like reflections from a wall or boundary. Because there is no loss of speed in a billiard, it is assumed that all collisions are elastic. Out of all the billiard table representations available, the Bunimovich stadium is discussed because it was the simplest billiard table which illustrated two dimensional nonlinear dynamics. There are many experiments trying to either prove quantum chaos' existence or to bridge the gap between classical and quantum chaos.

II. THE THREE TYPES OF CHAOS

The first, most commonly studied form of chaos is quantised chaos. Quantised chaos describes the quantisation of classically chaotic systems[2]. One quantises a classically chaotic system by taking the semi-classical limit of $\hbar \to 0$ or $m \to \infty$. When a chaotic system is quantised, the resulting quantum system is not necessarily chaotic, but affected nevertheless. This is shown by how a one-body atomic system behaves chaotically when treated classically. Semi-quantum chaos describes systems with both classical and quantum subsystems. One may express a semi-quantum chaotic system with a billiard with vibrating boundaries. A degree of vibration of a billiard is defined as the number of boundary dimensions that change with time. An example of a single degree of vibration quantum billiard would be a vibrating sphere. A more practical type of quantum billiard is the cylindrical billiard with vibrations along the radius and longitude with encompasses the quantum wire, which is discussed later. Lastly, there exists true quantum chaos. Systems exhibiting true quantum chaos are bounded, fully quantised and experience exponential sensitivity and repeat indefinitely. A system that experiences exponential sensitivity cannot have a fully discrete energy spectrum.

III. THE TRANSITION FROM $\mathbb{R}$ TO $\mathbb{R}^2$

Griffith's *Introduction to Quantum Mechanics* covers a lot of examples of one dimensional linear systems, the infinite square well being one of the examples covered [4]. In order to bridge the examples in class with two dimensional systems, a new tool has to be adopted: the phase space. A phase space is a space that represents all possible states of a system. In the case of the infinite square well, the phase space is a line. Another example involves an infinite circular well which is a two dimensional linear dynamical system, which had a phase space of a circular plane. It makes sense that the phase space has the same dimensions as the problem it is representing.

The well-known linear Schrödinger equation, which represents one dimensional non-chaotic systems is known to be:

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left[\frac{-\hbar^2}{2m}\nabla^2 + V\right]\Psi(x,t) \tag{1}$$

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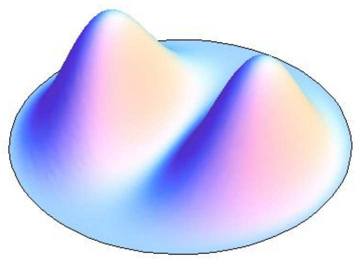

FIG. 1. An infinite circular well with an azimuthal quantum number $m = 1$ and $k = 1$.

The transition between one dimensional systems and two dimensional systems is an extra variable in the spatial gradient of the linear Schrödinger equation. If one is dealing with a two dimensional box, an extra variable y would be added to the spatial Laplacian. However, since the circular well is being studied, it is convenient to work in polar coordinates instead of in Cartesian coordinates. Suppose that $V(r) = 0$ when $r < a$, where a is the radius of the circular well, and $V(r) = \infty$ when $r > a$. These boundary conditions resemble the boundary conditions for the infinite square well. Because the circular well is symmetric, the wave function $\Psi(r, \theta)$ can be separated, similar to the separation of variables for stationary states, into its two spacial components $\Psi(r, \theta) = \gamma(r)\xi(\theta)$. Because the the wave function is separable, one can treat the infinite circular well as two, not necessarily independent, one dimensional problems. Using the boundary condition when $V(x) = 0$, and expressing the Laplacian ∇^2 in terms of r and θ [6], one can rewrite the Schrödinger equation as:

$$-\frac{\hbar}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\Psi(r, \theta) = E\Psi(r, \theta) \quad (2)$$

Substituting the separated wave function $\Psi(r, \theta) = \gamma(r)\xi(\theta)$, one finally arrives at equation [Eq. 3] where $k = \frac{\sqrt{2mE}}{\hbar}$ and μ is a separation constant similar to E.

$$\frac{d^2\gamma(r)}{dr^2} + \frac{1}{r}\frac{d\gamma(r)}{r} - \frac{\mu^2}{r^2}\gamma(r) = -k^2\gamma(r) \quad (3)$$

As with the infinite square well, the infinite circular well has only bound states and the angular solutions turn out to be $\xi_\mu(\theta) = \frac{1}{\sqrt{2\pi}}e^{i\mu\theta}$ for any $\mu \in \mathbb{Z}$. Derivations aside, one can see that the infinite circular well is very much the same as the infinite square well. This similarity will be useful when their billiard representations are discussed.

IV. THE BUNIMOVICH STADIUM

Although the infinite circular well is two dimensional, it does not convey chaos very well because it is still modelled by the linear Schrödinger equation. A new equation, for nonlinear dynamics, has to be introduced. Like the linear Schrödinger equation, the lesser-known nonlinear Schrödinger equation can be applied to both classical and quantum mechanical systems exhibiting chaos. Although the nonlinear Schrödinger equation is a classical field equation, it can be quantised to receive the quantum counterpart.

$$i\frac{\partial}{\partial t}\psi = \frac{1}{2}\nabla^2\psi + \kappa|\psi|^2\psi \quad (4)$$

It is important to note that the nonlinear Schrödinger equation does not describe the time evolution of its quantum states, unlike the linear Schrödinger equation. This is because the nonlinear Schrödinger equation models integrable systems. An integrable Hamiltonian system, specifically a quantum integrable system, has N independent constants of motion. As stated in the introduction, the Bunimovich stadium is a Hamiltonian system governed by Hamilton's equations. The stadium, in essence, is an idealisation of a game of billiards.

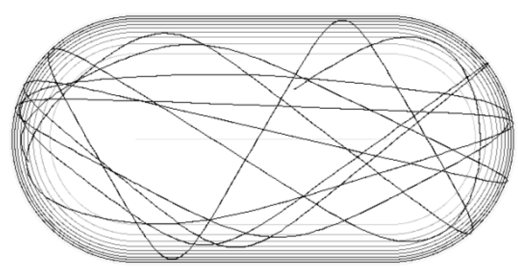

FIG. 2. The Bunimovich stadium with the trajectories of a particle[7].

V. MODELLING QUANTUM CHAOS USING BILLIARDS

Of all the examples aforementioned, all three situations could be expressed as billiard systems. The main advantage to using billiards to represent dynamical systems is that billiards capture all the dynamics of a Hamiltonian systems without having to determine its Poincaré map. Classically, the infinite square well can be thought of as a particle with mass bouncing between two barriers of infinite potential. Quantum mechanically, there exists standing waves that oscillate between the two barriers of infinite potential. Both cases are analogous to billiard systems in the sense that the billiards in the ad hoc billiard table bounce off of the ends of the table similar to the barriers of $V(x) = \infty$. The same could be said for the infinite circular well, a two dimensional equivalent, where a particle with mass bounces against the circumference of the circle where $V(x) = \infty$. A billiard corresponding to the two examples above is shown to have a circular shape in [Fig. 2]. The motion inside of the circular billiard is composed of straight lines. This is explained a priori, as a billiard table is composed of motion alternating from straight lines and mirror-like rebounds from its walls.

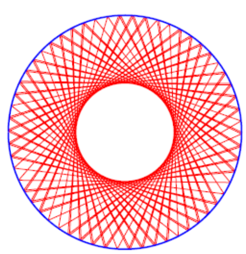

FIG. 3. Regular dynamics conveyed by the circular billiard. [1]

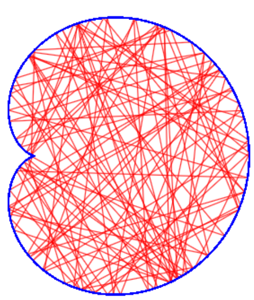

FIG. 4. Chaotic dynamics conveyed by the cardioid billiard. [1]

Note that in chaotic billiards, like in regular billiards, the motion within the billiards is still composed of straight lines. The slight eccentricity of the chaotic billiard causes a perturbation in the otherwise perfect motion of the billiard. In classical mechanics, motion can be considered chaotic if it is cannot be repeated. In other words, a particle could travel the same path N times ad infinitum. However, in the case of the chaotic billiard, it is easy to see that a particle would not *likely* travel the same path again after bouncing off the boundary. For the billiard to be chaotic in both classical and quantum dynamics, it must satisfy the following properties verbatim[8]:
1)it must be sensitive to initial conditions;
2)it must be topologically mixing;
3)its periodic orbits must be dense.

The first property is more commonly referred to as the butterfly effect. The butterfly effect simply states that each point in the billiard system is randomly affected by other points with different future paths; this effect is not immediately obvious from looking at figure 3. Topological mixing states that the billiard will evolve over time so that any region of its phase space will eventually overlap with any other region. The last property states that there are periodic orbits that move toward every point in the billiard from a random distance.

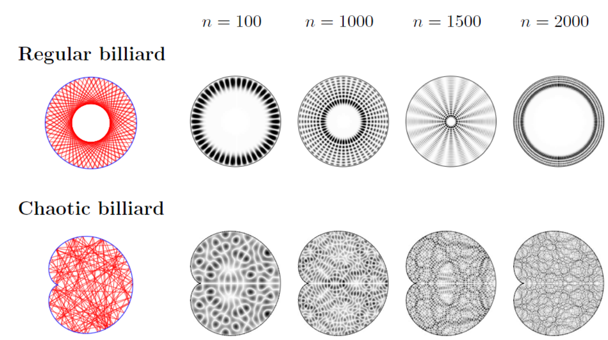

FIG. 5. The quantum, integrable billiards as the eigenstates n increases[1].

In figure 5, a regular and chaotic billiard are shown together as its eigenstates increase. The eigenstates are found to represent the structure of its corresponding classical dynamics. Eigenstates could be thought of as "pinned" positions of the particle.

VI. FUTURE ADVANCEMENTS IN QUANTUM CHAOS

The experiment performed by Shumpei Masuda, Shinichi Sawada and Yasushi Shimizu was studied. The experiment consisted of a quantum wire with fermions interacting with each other via a repulsive or attractive Coulomb interaction. A quantum wire is an adequately small, electrically conducting wire. One property of a quantum wire is that electrons will experience quantum confinement in the x and y directions. The experiment also used a hard wall potential in the z direction. Fermions are characterised by any particle which obeys Pauli's exclusion principle. Masuda et al used an effective Hamiltonian to reduce the complex three dimensional system to a pseudo one dimensional system. The classical dynamics of the system had a scaling property that the quantum counterpart did not have. However, Masuda et al calculated the Brody parameter for distributions of the nearest neighbour level spacing and found that the statistics of the energy level implies a corresponding scaling even in the quantum system [3].

In this study, Masuda et al used a modified maximum Lyapunov exponent instead of a normal maximum Lyapunov exponent to measure the chaotic irregularity in the classical system. A maximum Lyapunov exponent is defined mathematically by:

$$|\delta Z(t)| \approx e^{\lambda t} |\delta Z_0| \tag{5}$$

where λ is the Lyapunov exponent and $\delta Z(t)$ is the separation of infinitesimally close trajectories of the particles in the system. The system described by equation 5 is said to be chaotic if the Lyapunov exponent λ is positive. The difference between the old maximum Lyapunov exponent, which will be referred to as MLE henceforth, was that it measured the rate per unit of the time for separation and the new MLE measured one per unit of the separation.

Masuda et al also claimed that the area of a chaotic region in a Poincaré map does not necessarily determine the system's chaotic irregularity despite most authors saying otherwise. This change in method allowed Masuda et al to compare the chaotic irregularity of classical orbits with varying energies.

In addition to the experiment involving the quantum wire, a fibre laser also depends on the concept of quantum chaos and more specifically: quantum billiards.

VII. CONCLUSION

Quantum chaos was manifested easily in billiards, and many dynamical systems can be represented as billiards. It was shown that the one dimensional and two dimensional cases of the infinite square and circular well could both be represent by regular, non chaotic billiards. The Bunimovich stadium was also found to represent classical and quantum nonlinear dynamical systems that exhibited chaos. Regular billiards were represented by a circular billiard while chaotic billiards were represented by a regular billiard under some perturbation. Three types of quantum chaos were also introduced, quantised chaos being the most studied. Classical and quantum chaos were found to have a close correspondence from Masuda et al's experiment involving a two-spinless-fermion system in a quantum wire. It also followed that if a classical system is chaotic, its quantum counterpart was as well from the analysis of corresponding billiard systems. The debate about chaos theory and the existence of quantum chaos is still ongoing. For the most part, quantum chaos remains a worthwhile pursuit and a confusing field of study even today.

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